Return of the Σ^I:

Proof of Thm II.8

For each $r \in \mathbb{N}$ $\quad S^r := \{ \sigma \in J^1(M,N) \mid \text{rank}(\sigma) = r \}$

$\text{rank}(\sigma)$ is rank $(df(x))$ where $f \in C^\infty(M,N)$ represents $\sigma \in J^1(M,N)_{x,y}$

is smooth subm of $J^1(M,N)$, codim $S^r = (n-r)(m-r)$

and $$\underbrace{\Sigma^i(f)}_{\text{pts in } M \text{ where corank}(f) = i} = \begin{cases} (j^1 f)^{-1}(S^{m-i}) & \text{if } m \geq n \\ (j^1 f)^{-1}(S^{n-i}) & \text{if } m \leq n \end{cases}$$

By Thm 4, for generic f we have $j^1 f \pitchfork S^r$ and thus by Prop. II.11 $\Sigma^i(f)$ is a subm of M of codimension $i(|n-m|+i)$. ∎

! $j^1 f \pitchfork S^r$ $\underline{\underline{\forall r}}$, such f are called one-generic or good. Their existence follows also from Thm. 4 or by a generalization of Thm 4 to stratified sets / submfs.

Another extension considers multijet spaces (useful for injectivity & self-intersections of maps).

Here an r-fold k-jet bundle consists of the following data:

Let $M^r = \underbrace{M \times \dots \times M}_{r}$ and

$$M^{(r)} := \mathrm{Conf}_r(M) = \{ (x_1, \dots, x_r) \in M^r \mid x_i \neq x_j \ \forall i \neq j \}$$

For $s : J^k(M,N) \to M$ source map set

$$s^r : J^k(M,N)^r \to M^r \quad \text{and}$$

$$J^k_r(M,N) := (s^r)^{-1}(M^{(r)}) \subset J^k(M,N)^r$$

For $f : M \to N$ smooth $j^k_r f : M^{(r)} \to J^k_r(M,N)$ is given by $j^k_r f(x_1, \dots, x_r) = (j^k f(x_1), \dots, j^k f(x_r))$.

8. Theorem

M, N mfs, $S \subset J^k_r(M,N)$ submf. Let

$$\mathcal{T}_S = \{ f \in C^\infty(M,N) \mid j^k_r f \pitchfork S \}. \text{ Then}$$

$\mathcal{T}_S$ is a residual subset of $C^\infty(M,N)$.

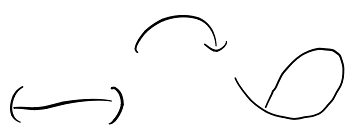
If S compact, then $\mathcal{J}_S$ is open in $C^\infty(M,N)$.

Let us quickly consider the case $m < n$, where we can now prove <u>Whitney's immersion and (easy) embedding theorems</u>.

Recall, $f \in C^\infty(M,N)$ is an

- immersion if df_x injective for all $x \in M$
- embedding if
  - f immersion & homeom. onto its image — in general
  - f immersion & injective — M compact

<u>e.g.</u>

-  imm., not an embedding
- $\mathbb{RP}^2 \to \mathbb{R}^3$

  Roman surface, not immersion

  Boy's surface, imm. not an embedding

  ($\rightsquigarrow$ Wiki)

Let now $m<n$, so $\operatorname{corank}(df) = m - \operatorname{rank}(df)$
$= \dim \ker(df)$

9. Lemma $f: M \to N$ imm. $\Leftrightarrow$ $j^1 f(M) \cap \bigcup_{r \neq m} S^r = \emptyset$

Proof: Clear ∎

10. Lemma $\operatorname{Imm}(M,N) = \{ f \in C^\infty(M,N) \mid f \text{ immersions} \}$
is open in $C^\infty(M,N)$.

Proof: $S^m \subset J^1(M,N)$ open and
$\operatorname{Imm}(M,N) = M(S^m)$, hence open in
the Whitney topology.
∎

11. Theorem (Whitney)

If $n \geq 2m$, then $\operatorname{Imm}(M,N)$ is an open dense subset of $C^\infty(M,N)$ (in the Whitney C^∞-topology).

Proof: • Open: Lemma 10

- Dense:

For $0 \leq r < m$ we have

$$\operatorname{codim} S^r = \underbrace{(m-r)}_{\geq 1} \cdot \underbrace{(n-r)}_{\geq 2m-r} \qquad \left(\begin{array}{l}\text{corank} \\ \text{product formula}\end{array}\right)$$

$$\geq 2m-r \geq m+1$$

In these dimensions

$$j^1 f \pitchfork S^r \iff j^1 f(M) \cap S^r = \emptyset$$

Lemma 9: $f \in \operatorname{Imm}(M,N) \iff j^1 f \pitchfork S^r \ \forall r < m$

Thm 4: $\operatorname{Imm}(M,N)$ is a residual subset of $C^\infty(M,N)$, hence dense. $\blacksquare$

12. Theorem (Whitney)

Let M be compact. If $n \geq 2m+1$, then the set of embeddings $M \to \mathbb{R}^n$ is residual in $C^\infty(M,\mathbb{R}^n)$.

Proof:

M compact $\Rightarrow$ f embedding $\iff$ f immersion & injective

To show: $\operatorname{Inj}(M,N) \subset C^\infty(M,N)$ is residual:

$f: M \to N$ injective if and only if

for $j_2^0 f : \underset{\{(x_1,x_2)\in M^2 \mid x_1 \neq x_2\}}{\underset{\shortparallel}{M^{(2)}}} \longrightarrow J_2^0(M,N) \subset (M\times N)\times(M\times N)$

and $S := (t^2)^{-1}(\underbrace{\Delta_N}_{=\{(y,y)\mid y\in N\}\subset N^2})$

it holds

$$j_2^0 f(M^{(2)}) \cap S = \emptyset .$$

$t^2 : J_2^0(M,N) \to N^2$ is a submersion, so $S \subset J_2^0(M,N)$ is a submf. Moreover,

$\text{codim } S = \text{codim } \Delta_N = n > 2m = \dim M^{(2)}$,

so that

$$j_2^0 f \pitchfork S \iff j_2^0 f(M^{(2)}) \cap S = \emptyset .$$

Thus, f injective if and only if $j_2^0 f \pitchfork S$ and the result follows by multijet transversality. $\blacksquare$

Remark: The set of embeddings $M \to \mathbb{R}^n$ is also open, but this is a bit harder to prove...